Applications of holomorphic curves to symplectic topology via Hamiltonian dynamical systems

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Overview

In these lectures we will discuss several applications of holomorphic curves to symplectic topology. In particular, we will develop and apply tools from Hamiltonian Floer theory which couples the theory of holomorphic curves to techniques from Morse theory. Hamiltonian dynamical systems, in particular their periodic orbits, will play a prominent role in this story.

Lectures

1. Lecture 1. Floer trajectories
2. Lecture 2. A rapid introduction to Morse homology
3. Lecture 3. From Morse homology to Floer homology
4. Lecture 4. Applications: Periodic orbits and nonsqueezing

Lecture 1: Floer trajectories

Starting from holomorphic spheres we introduce the notion of a Floer trajectory. We discuss their relation to the periodic orbits of Hamiltonian dynamical systems as well as their basic properties (compactness, transversality, etc.)

A recurring theme in this lecture will be that Floer trajectories generalize the notion of trajectories of (negative) gradient vector fields of functions on manifolds.

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- X_H, the time-periodic vector field determined by Hamilton's equation $i_{X_H}\omega = -dH_t$.

- $\phi_H^t \colon M \to M$, the time-t flow of X_H (defined for all $t \in \mathbb{R}$).

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$$\begin{cases} \dot{x}_j & = +\frac{\partial H}{\partial y_j}(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \\ \dot{y}_j & = -\frac{\partial H}{\partial x_j}(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \end{cases}$$

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Observation

If H does not depend on t, then every critical point of H is a zero of X_H and hence a fixed point of ϕ_H^t for all $t \in \mathbb{R}$.

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Theorem (Poincaré recurrence)

For almost every point $p \in M$ there is a sequence of times $t_j \nearrow \infty$ such that

$$\phi_H^{t_j}(p) = p.$$

Periodic orbits

Definition

A smooth map $x\colon \mathbb{R} \to M$ is a period orbit of H (or X_H) with period T if for all $t \in \mathbb{R}$ we have $x(t+T) = x(t)$ and

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A T-periodic orbit $x(t)$ of H is said to be **nondegenerate** if the linear map $(d\phi_H^T)_{x(0)}\colon T_{x(0)}M \to T_{x(0)}M$ does not have one as an eigenvalue.

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Fact

A nondegenerate T-periodic orbit is isolated away from all other T-periodic orbits.

Example: Let $f : M \to \mathbb{R}$ be a smooth function.

Let p be a critical point of f.

Removing another point we get a map $u\colon \mathbb{R} \times S^1 \to M$ such that

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0,$$

where $e^{2\pi(s+it)} = x + iy \in \mathbb{C} \setminus \{0\} \simeq \mathbb{R} \times S^1$.

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Definition

A **Floer trajectory** is a smooth map $u\colon \mathbb{R}\times S^1 \to M$ which satisfies the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0. \tag{1}$$

Here, J_t is a smooth S^1-family of compatible a.c.s.'s on (M,ω) and H is a smooth function on $S^1\times M$.

Example: Let $f\colon M \longrightarrow \mathbb{R}$ be a smooth function.

Let $J_t = J$ be t-independent and consider a Floer trajectory $\gamma\colon \mathbb{R} \to M$, which also does not depend on t. Then, by (1) we have:

$$\dot{\gamma} - J(\gamma)X_f(\gamma) = 0.$$

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Here, ∇f is the gradient vector field of f with respect to the metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$, which is defined by the equation

$$g_J(\nabla f, \cdot) = dH(\cdot). \tag{3}$$

Asymptotic behavior

Define the **energy** of a Floer trajectory u by

$$E(u) = \int_{\mathbb{R}\times S^1} \omega(u)\left(\frac{\partial u}{\partial s}, J_t(u)\frac{\partial u}{\partial s}\right) ds\, dt.$$

Proposition

Suppose that all of the 1-periodic orbits of H are nondegenerate. Then for every Floer trajectory u with $E(u) < \infty$, the limits

$$u(\pm\infty) = \lim_{s\to\pm\infty} u(s,t)$$

exist in $C^\infty(S^1, M)$ and are 1-periodic orbits of H.

Example: Let $f\colon M \longrightarrow \mathbb{R}$ be a smooth function.

Suppose f is a Morse function and $J_t = J$ is t-independent. Let $\gamma\colon \mathbb{R} \to M$ satisfy

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Fact

Then we have

$$E(\gamma) = \int_{\mathbb{R}} |\dot{\gamma}(s)|^2_{g_J}\, ds < \infty,$$

and the limits $\gamma(\pm\infty) = \lim_{s\to\pm\infty} \gamma(s)$ *exist and are critical points of* f*.*

Organizing Floer trajectories

From now on we will only be interested in the set $\mathcal{P}(H)$ of contractible 1-periodic orbits of H, and the Floer trajectories which are asymptotic to elements of $\mathcal{P}(H)$. For $x, y \in \mathcal{P}(H)$ set

$$\mathcal{M}(x,y) = \left\{ u \colon \mathbb{R} \times S^1 \to M \;\middle|\; \begin{array}{c} u \text{ solves } (1), \\ u(-\infty) = x(t), \\ u(+\infty) = y(t). \end{array} \right\}$$

If all the elements of $\mathcal{P}(H)$ are nondegenerate, we call H a **Floer Hamiltonian**. In this case all the Floer trajectories of interest lie in one of the spaces $\mathcal{M}(x,y)$.

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   (This is a very strict assumption but includes the important example of symplectic tori, $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.)
3. H is a Floer Hamiltonian.

   (This is a generic property in the space $C^\infty(S^1 \times M)$.)

The first and third assumptions imply that $\mathcal{P}(H)$ is a finite set, and hence there are finitely many spaces $\mathcal{M}(x, y)$.

Implications of the second assumption.

The assumption that $\pi_2(M)$ is trivial implies that any holomorphic sphere has zero symplectic area and hence zero energy. Hence, there is no bubbling for sequences of holomorphic curves.
This assumption also allows us to associate two important quantities to each $x \in \mathcal{P}(H)$:

The first of these is the **action** of $x \in \mathcal{P}(H)$ defined by

$$\mathcal{A}_H(x) = \int_{S^1} H(t, x(t))\, dt - \int_{D^2} \bar{x}^* \omega,$$

where $\bar{x}$ is a smooth map from the unit disc $D^2 \subset \mathbb{C}$ to M such that $\bar{x}(e^{2\pi i t}) = x(t)$.

The second quantity is the **Conley-Zehnder index** of an orbit $x \in \mathcal{P}(H)$. This is an integer $\mathrm{CZ}(x)$ which is defined in the spirit of the Maslov class.

Nonstandard normalization: if x is a local maximum (minimum) of a C^2-small Morse function f, then $\mathrm{CZ}(x) = \frac{1}{2}\dim M$ $(-\frac{1}{2}\dim M)$.

Properties of $\mathcal{M}(x;y)$: Transversality

Let $\mathcal{J}_{S^1}$ be the space of smooth S^1-families of compatible a.c.s.'s on $(M;\omega)$.

Theorem (Floer, Hofer, Salamon)

For every H there is a subset $\mathcal{J}_{reg}(H) \subset \mathcal{J}_{S^1}$ of second category such that each for $J_t \in \mathcal{J}_{reg}(H)$, the corresponding spaces $\mathcal{M}(x;y)$ are smooth finite dimensional manifolds of dimension $CZ(x) - CZ(y)$.

Note that for $x \neq y$, there is a free $\mathbb{R}$-action on $\mathcal{M}(x;y)$ by translation; $\tau \cdot u(s;t) = u(s+\tau;t)$. Set

$$\hat{\mathcal{M}}(x;y) = \mathcal{M}(x;y)/\mathbb{R}.$$

Properties of $\mathcal{M}(x, y)$: Compactness and Gluing

For a generic choice of J_t, the moduli (quotient) space $\widehat{\mathcal{M}}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ is a smooth manifold of dimension $\mathrm{CZ}(x) - \mathrm{CZ}(y) - 1$.

Definition

A k**-times broken** Floer trajectory from x to y is a collection of $k+1$ classes $u_j \in \widehat{\mathcal{M}}(z_j, z_{j+1})$ such that $z_0 = x$, $z_{k+1} = y$ and

$$\mathrm{CZ}(x) > \mathrm{CZ}(z_1) > \cdots > \mathrm{CZ}(z_k) > \mathrm{CZ}(y).$$

Note that $k \leq \mathrm{CZ}(x) - \mathrm{CZ}(y) - 1$.

Denote the set of k-times broken Floer trajectory from x to y by $\partial_k \mathcal{M}(x, y)$.

Compactness and Gluing continued

Theorem (Gromov, Floer)

For $J_t \in \mathcal{J}_{reg}(H)$, the moduli space $\widehat{\mathcal{M}}(x, y)$ has a natural compactification as a smooth $(\mathrm{CZ}(x) - \mathrm{CZ}(y) - 1)$-dimensional manifold with corners. The strata of codimension k in the compactification is $\partial_k \mathcal{M}(x, y)$.

Example

If $\mathrm{CZ}(x) = \mathrm{CZ}(y) + 1$ then $\widehat{\mathcal{M}}(x, y)$ is a compact 0-dimensional manifold.

Example

If $\mathrm{CZ}(x) = \mathrm{CZ}(y) + 2$ then the compactification of $\widehat{\mathcal{M}}(x, y)$ is a 1-dimensional manifold with boundary given by

$$\cup_{\mathrm{CZ}(z)=\mathrm{CZ}(x)-1} \widehat{\mathcal{M}}(x, z) \times \widehat{\mathcal{M}}(z, y).$$

Remarks on the proof of compactness and gluing